Sets, Cartesian Products, Relations and Functions

Sets

A set is a collection of distinct, related items (known as elements). It may be empty or non-empty. Since the elements in a set may be presented in any order without changing the value of a set, sets with several elements may be presented in many ways without changing the value of a set.

Sets are not only essential building blocks in several areas of mathematics – functions, relations, statistics and probability – but are also evident when using applications software. For example, a database table is a set since it is a collection of distinct, related elements. The elements in a database table are typically combinations of various values. For example, in the DEPT table in figure 1, an element has a department number, a department name and a location.

dept		
DEPTNO	**DNAME**	**LOC**
1	ACCOUNTING	LONDON
2	RESEARCH	YORK
3	SALES	BIRMINGHAM
4	OPERATIONS	LEEDS

Figure 1: DEPT table

We may represent this table as the following set.

{ (1, ACCOUNTING, LONDON) ,

(2, RESEARCH, YORK) ,

(3, SALES, BIRMINGHAM) ,

(4, OPERATIONS, LEEDS) }

Note how each element of the set is presented within parentheses with the components separated by commas. These elements are known as tuples or, more specifically in this case, 3-tuples. A 3-tuple – a tuple with three components – may be referred to as a triple. The main differences between tuples and sets are:

- tuples have a fixed size whereas sets have variable sizes
- the components of a tuple may vary in type – such as integers and text – whereas a set has values of a single type

- the components of a tuple are presented in a particular order and thus (1,2) is different to (2,1), whereas the elements of a set may be presented in any order.

The example we used to illustrate the importance of ordering in a tuple should look very familiar. You have been used to presenting coordinates as a tuple since these were introduced to you at school. In figure 2 we present a graph with the tuples (coordinates) (1,2) and (2,1) highlighted.

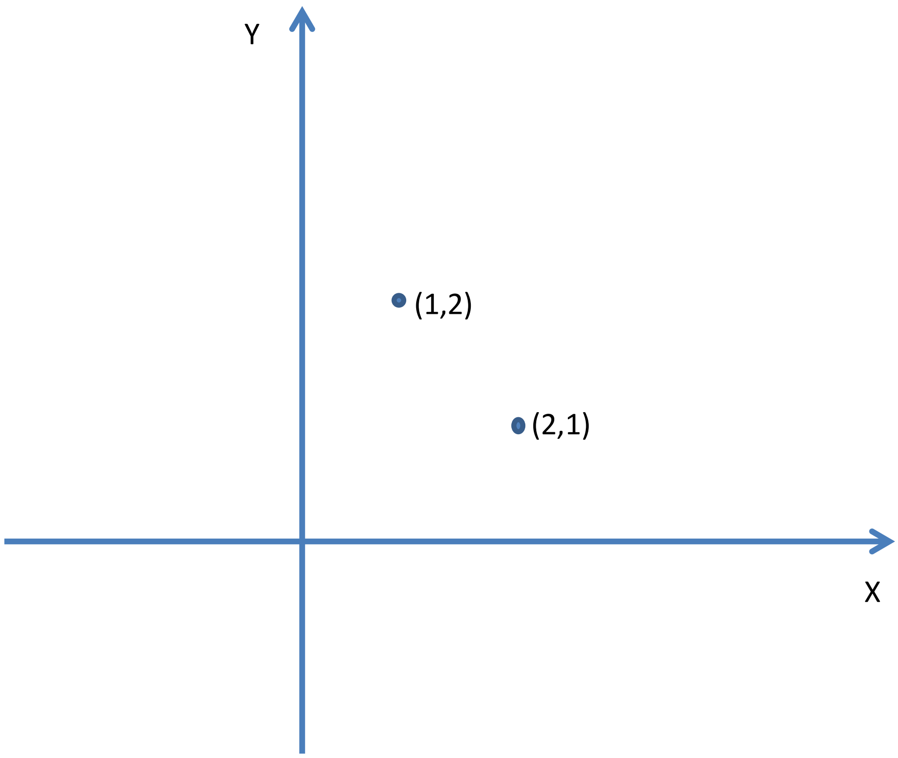


Figure 2: Graph with Coordinates

You can actually represent any point on the graph as a coordinate (2-tuple). We typically refer to a 2-tuple as a pair or, more specifically, an **ordered pair**.

Cartesian Product…

We can generate a set of ordered pairs by using the cartesian product operator X. This is named after the French mathematician and philosopher René Descartes. The Cartesian product of two sets is:

the set of ordered pairs where the first element comes from the first set and the second element comes from the second set.

For example, if we have the set A = {1,2,3} and the set B = {4,5} then the Cartesian product of A and B, written A X B is:

{(1,4), (1,5), (2,4), (2,5), (3,4), (3,5)}

Note that this set has 6 elements which is the product of the number of elements in each of the input sets.

The Cartesian product of B and A, written B X A is:

{(4,1), (4,2), (4,3), (5,1), (5,2), (5,3)}

Note that unlike multiplication with numbers, the product of two sets is typically different if we reorder the inputs. That is, B X A ≠ A X B.

Given that P = {2,3,5,7,11}, E = {-4,-2,0,2,4} and S = {0,1,4,9} evaluate the following Cartesian product expressions:

1. P X E
2. E X P
3. S X E
4. S X P
5. P X E X S

Now if we let **R** represent the real numbers (the set of any number you can think of) then the Cartesian product of **R** with itself, **R** X **R** (or $\mathbf{R}^2$) is the set of all possible points on a graph. That is, any point in the graph presented in figure 2 will be an element of **R** X **R**.

Relations…

Let A be the set {1,2,3,4,5,6,7,8,9,10}. Then A X A (or A^2) is the set:

{(1,1),(1,2),(1,3)…,(10,8),(10,9), (10,10)}

That is, it is the set of 100 ordered integer pairs between (1,1) and (10,10).

Now, if we take the following subset of A^2:

{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3),(5,1),(5,2),(5,3),(5,4),(6,1),(6,2),(6,3),(6,4),(6,5),(7,1),(7,2),

(7,3),(7,4),(7,5),(7,6),(8,1),(8,2),(8,3),(8,4),(8,5),(8,6),(8,7),(9,1),(9,2),(9,3),(9,4),(9,5),(9,6),

(9,7),(9,8),(10,1),(10,2),(10,3),(10,4),(10,5),(10,6),(10,7),(10,8),(10,9)}

you should notice that all of these pairs have something in common. That is, they all exhibit a common relation between their first and second values. In each case the first value is greater than the second value. Thus we have defined the relation **less than** between the set of integers from 1 to 10 and itself.

Provide a sensible name for each of the following relations:

1. {(1,1),(2,1),(2,2),(3,1),(3,3),(4,1),(4,2),(4,4),(5,1),(5,5),(6,1),(6,2),(6,3),(6,6),(7,1),(7,7),(8,1),(8,2),(8,4),(8,8),(9,1),(9,3),(9,9),(10,1),(10,2),(10,5),(10,10)}
2. {(1,1),(2,4),(3,9)}
3. {(1,1),(1,3),(1,5),(1,7),(1,9),(2,2),(2,4),(2,6),(2,8),(2,10),(3,1),(3,3),(3,5),(3,7),(3,9),(4,2),(4,4),(4,6),(4,8),(4,10), (5,1),(5,3),(5,5),(5,7),(5,9),(6,2),(6,4),(6,6),(6,8),(6,10), (7,1),(7,3),(7,5),(7,7),(7,9),(8,2),(8,4),(8,6),(8,8),(8,10), (9,1),(9,3),(9,5),(9,7),(9,9),(10,2),(10,4),(10,6),(10,8),(10,10)}

Write out the set of order pairs that represent the following relations:

1. double
2. number of factors
3. at least 2 greater than

So a relation may be defined as a **subset of the cartesian product of two sets**.

Another was of defining a relation is as a:

Collection of associations between two or more sets.

This may be represented as a set of ordered pairs or in pictorial form in a sets and arrows diagram as in figure 3. Each association between an element of the input set (on the left) and an element of the output set (on the right) is represented by an arrow between the two elements.

Here we present the relation **greater than** between the set {1,2,3,4} and the set {1,2,3,4,5}.

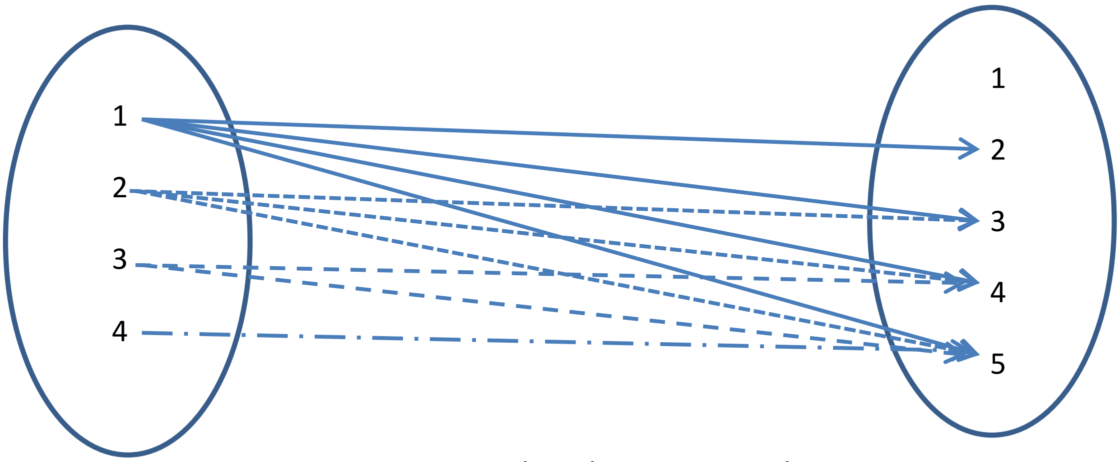
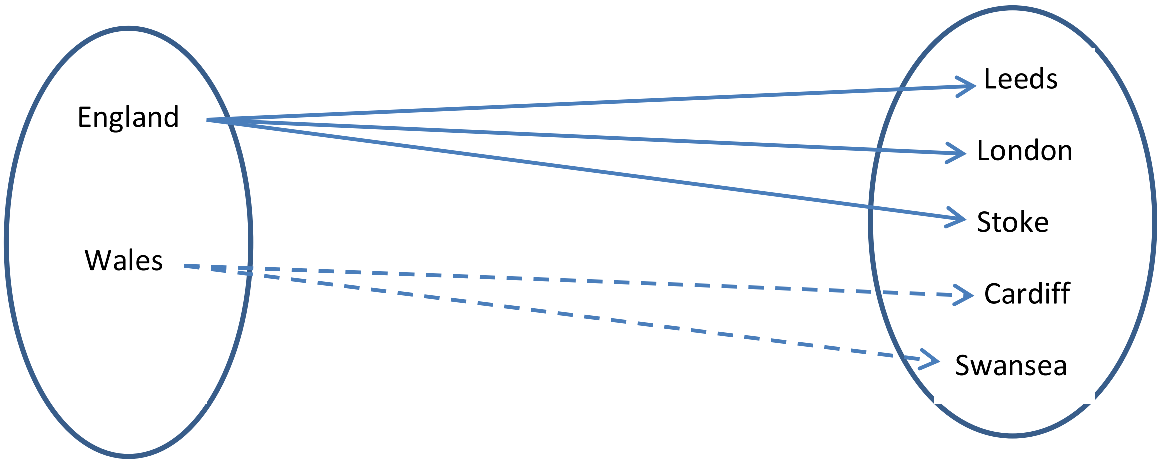


Figure 3: The relation greater than

Of course relations do not have to be numeric. We may have the relation **city in** which associates a country with cities in that country. We present an example of this relation in figure 4.



Thus a relation maps elements of one set to another set.

Draw sets and arrows diagrams to represent each of the following relations (you may select relevant input and output sets):

1. brother of (input set is subset of students in this class)
2. mobile phone make (input set is subset of students in this class)
3. age (input set is subset of students in this class)
4. actor in film (input set is a set of films)

There are four main categories of relations:

- many to many
- one to many
- many to one
- one to one

These categories reflect the number of associations between elements of the input set and elements of the output set. The relation less than presented in figure 3 is an example of a many to many relation, where many elements in the input set are associated with many elements of the output set. That is, many elements in the input set are associated with the same element in the output set and there are elements in the input set that are associated with more than one element in the output set. For example, the numbers 1, 2, 3 and 4 are all associated with the number 5, and the number 1 is associated with the numbers 2,3,4 and 5.

The relation city in presented in figure 4 is an example of a one to many relation where the elements in the input set may be associated with more than one element in the output set but no two elements in the input set are associated with the same element.

A many to one relation has more than one element of the input set associated with the same element of the output set but each element of the input set is associated with exactly one element of the output set. An example relation is **is studying** where students on this module are associated with the degree they are studying.

A one to one relation associates each element of the input set with a different element in the output set. An example relation is **has KU ID number** which associates each student on this module with their KU ID number.

Categorise each of the following relations as: many to many; many to one; one to many or one to one.

1. brother of (input set is subset of students in this class)
2. mobile phone make (input set is subset of students in this class)
3. age (input set is subset of students in this class)
4. actor in film (input set is a set of films)
5. square root (input and output sets are real numbers)
6. KU ID number (input set is students in this class)
7. module studying (input set is students in this class and output set is set of level 4 modules)

Functions…

The last two categories of relations actually have their own name – functions. A function is a mapping from an input set to an output set where elements of the input set are associated with exactly one element of the output set. That is, if we were to represent a function pictorially using the sets and arrows notation, then no element of the input set would have more than one arrow coming from it.

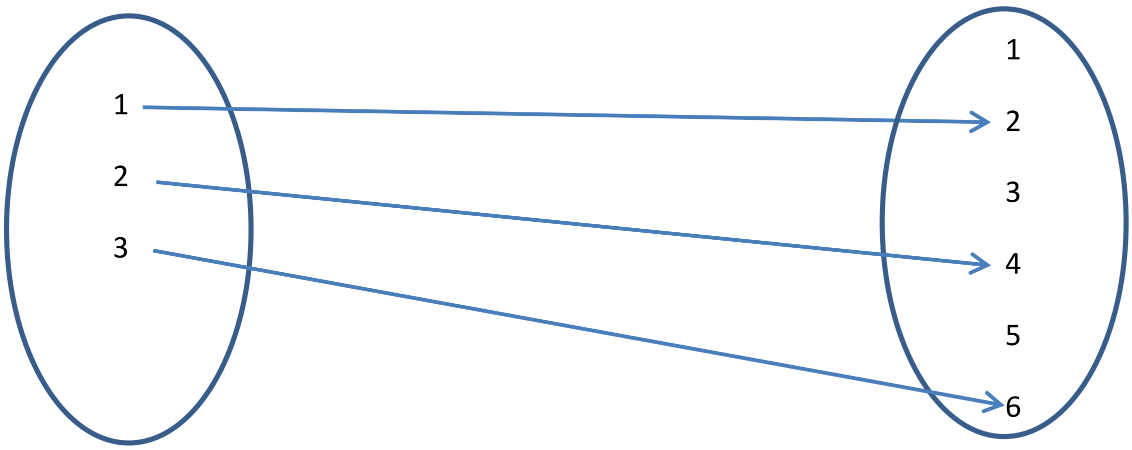
Functions are ubiquitous in the world of mathematics, applications software and business. You all have a wealth of experience of using functions. At school you would have seen functions defined as in the following example:

$$f(x) = 2x$$

Here each input (represented by the variable x) is doubled to produce the output. This may be represented by the following set of ordered pairs:

{…(1,2),(2,4),(3,6)…}

or by the following sets and arrows diagram:

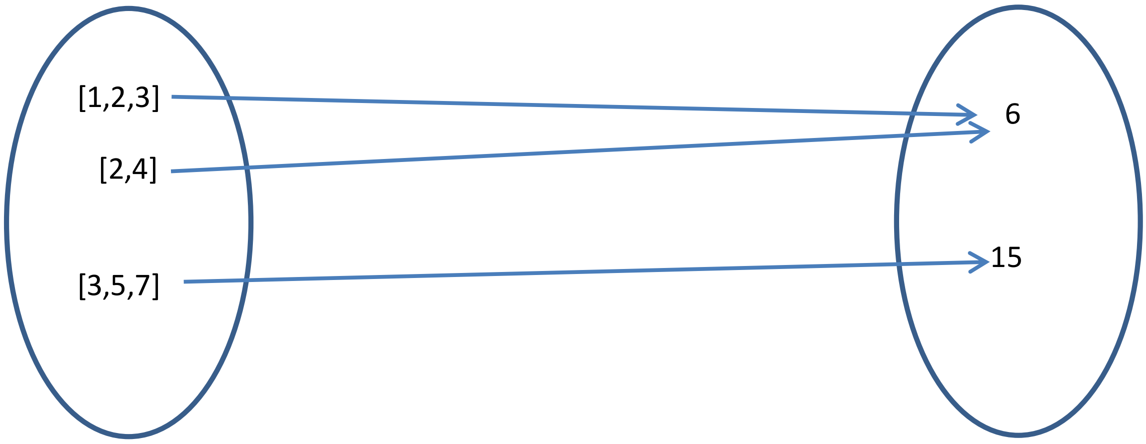


You have also some experience of using the functions provided by applications software. For example, Excel provides the =SUM function which takes a collection of numbers as input and returns a single number which is the sum of the input numbers.

This may be represented by the set of ordered pairs:

{…([1,2,3],6),([2,4],6),([3,5,7],15)…}

where we use [1,2,3] to represent the numbers input into the sum function. We can represent this in a sets and arrows diagram as follows:



Functions are used everywhere in everyday life. The sum function is used to generate the cost of a meal or of shopping. Television listings use functions to determine the start and finish time of programmes. Banks use multiplication to determine the interest paid on accounts. When you login to the Kingston University system a function is used to test whether you have input a valid username/password pair.

What all of these functions have in common is that given a particular input they return only one possible output. That is, unlike relations that may return many outputs, a function many only return a single output.

Which of the relations listed in the previous set of questions are functions?

The formal definition of a function requires three components:

- the input set (DOMAIN)
- the output set (CODOMAIN)
- the rule that associates elements in the input set with elements in the output set

Here are some example functions…

Example Function 1

Input set: Set of students in this class

Output Set: Set of months of the year

Rule: month of birth

Example Function 2

Input set: Set of integers

Output Set: Set of integers

Rule: square

Example Function 3

Input set: Set of KU academic staff

Output Set: Set of KU ID numbers

Rule: ID number

Define four functions only one of which may be a numeric function.

Every function may be categorised as many to one or one to one. A many to one function associates at least two input values with the same output value. A one to one function associates each input with a different output.

Thus, *month of birth* is many to one (many students will share the same month of birth), *square* is many to one (given the domain is the integers which includes negative and positive values) and *ID number* is one to one (since each member of academic staff will have a different ID number).

For each of your four functions state whether it is many to one or one to one.
